UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATH3506**

ASSESSMENT PATTERN : **MATH3506A**

MODULE NAME : **Mathematics in Biology I**

DATE : **30-Apr-09**

TIME : **14:30**

TIME ALLOWED : **2 Hours 0 Minutes**

2008/09-MATH3506A-001-EXAM-3

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Births in a population of long-lived individuals occur randomly and independently in time. The probability that an individual gives birth in the small time interval $[t, t + \delta t)$ is $b(t)\delta t + O(\delta t^2)$ where $b(t) \geqslant 0$.

   (a) Show that the probability $P(t)$ that an individual does not give birth in $[0, t)$ is given by
   $$P(t) = \exp\left(-\int_0^t b(s)\, ds\right).$$

   (b) Show that provided $tP(t) \to 0$ as $t \to \infty$ the expected time for an individual to first give birth is
   $$\bar{T} = \int_0^\infty P(t)\, dt.$$

   Now suppose that $b(t) = te^{-\lambda t}$.

   (c) What is the probability that an individual will give birth sometime?

   (d) Given that an individual gives birth sometime, what is the probability that they do not give birth in the time interval $[0, t)$?

   [You may ignore deaths].

2. An fish population is modelled by the discrete time system
   $$x_{t+1} = f(x_t), \quad t = 0, 1, 2, \ldots \qquad (1)$$
   where f is a smooth real-valued function defined on $[0, \infty)$.

   (a) Write down the equation that x^* must satisfy if it is a steady state population of (1). What is the condition that a steady state x^* is locally asymptotically stable?

   (b) When $f(x) = \dfrac{rx}{1 + x^2}$, where $r > 0$, sketch the cobweb map for (1) for the cases $r < 1$ and $r > 1$.

   (c) The fish population is now harvested, so that $f(x) = \dfrac{rx}{1 + x^2} - x$. Sketch the cobweb maps for the cases $8/3 < r < 4$ and $r > 4$ and comment briefly on your results.

3. A food chain of 3 species with densities x_1, x_2, x_3 is modelled via the Lotka-Volterra equations

$$\begin{aligned} \frac{dx_1}{dt} &= x_1(r_1 - a_{11}x_1 - a_{12}x_2) \\ \frac{dx_2}{dt} &= x_2(-r_2 + a_{21}x_1 - a_{22}x_2 - a_{23}x_3) \\ \frac{dx_3}{dt} &= x_3(-r_3 + a_{32}x_2 - a_{33}x_3) \end{aligned} \tag{2}$$

where $r_i > 0$ for $i = 1, 2, 3$ and $a_{ij} > 0$ for indices i, j where it is defined.

(a) Give a brief explanation of the model.

(b) State the Lyapunov stability theorem.

(c) Assuming that the system (2) has a unique interior steady state x^*, and by considering a function V of the form

$$V(x) = \sum_{i=1}^{3} \gamma_i \left\{ x_i - x_i^* - x_i^* \log\left(\frac{x_i}{x_i^*}\right)\right\},$$

for suitable $\gamma_i > 0$ $(i = 1, 2, 3)$, show that if the initial populations $x_i(0) > 0$ for $i = 1, 2, 3$ then the solution of (2) tends to the interior steady state x^* as t tends to infinity.

4. In an age-structured population there are n age classes and the density at age k at time t is $N_k(t)$. $n > 0$ is the maximum age of any individual. The expected number of offspring to females of age k is b_k for $k = 1, \ldots, n$ and the probability that an individual of age k, where $0 \leqslant k \leqslant n-1$ (with $k = 0$ newborns), survives to age $k+1$ is p_k.

(a) Show that $N(t+1) = LN(t)$ where $N(t) = (N_1(t), \ldots, N_n(t))$ and L is some $n \times n$ real matrix which you should find.

(b) Derive the Euler-Lotka equation for the eigenvalues of L, and show that L has a unique positive eigenvalue λ_0.

(c) In the case that $b_k = 0$ for $k = 1, \ldots, n-1$ and $b_n = b > 0$, show that

$$N(t+n) = \omega N(t), \quad t = 0, 1, \ldots,$$

where $\omega > 0$ is a constant which you should find. Hence determine how the age *distribution* of the population changes qualitatively with time.

5. A model for the interaction of two species of densities x_1, x_2 is given by

$$\begin{aligned}\frac{dx_1}{dt} &= r_1 x_1\left(1-\frac{x_1}{K_1}\right) - c_1 x_1 x_2\\ \frac{dx_2}{dt} &= r_2 x_2\left(1-\frac{x_2}{K_2}\right) - c_2 x_1 x_2,\end{aligned} \tag{3}$$

where $r_1 > 0, r_2 > 0, K_1, K_2 > 0$ and $c_1, c_2 > 0$.

(a) What kind of interspecies interaction does the system (3) model?

(b) Rewrite the system (3) in the form

$$\begin{aligned}\frac{du_1}{d\tau} &= u_1(1-u_1-a_{12}u_2)\\ \frac{du_2}{d\tau} &= ru_2(1-u_2-a_{21}u_1),\end{aligned} \tag{4}$$

where $\tau = r_1 t$, $u_1 = x_1/K_1$, $u_2 = x_2/K_2$, and r, a_{12}, a_{21} are parameters which you should find.

(c) Determine all possible steady states of (4) and characterise the local stability of any steady state that lies in the interior of the first quadrant.

(d) Sketch the phase plane for (4) in the case $a_{12} > 1, a_{21} > 1$.